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by R. V. Gamkrelidze

-USSR-

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FOREWORD

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THE THEORY OF OPTIMAL FAST-ACTING PROCESSES IN LINEAR SYSTEMS

-USSR-

[Following is a translation of an article by R. V. Gamkrelidze and presented by the Academician A. A. Dorodnitsyn, in Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya (The News of the Academy of Sciences of the USSR, Mathematical Series), Vol 22, No 4, Moscow, 1958, pages 449-474.]

In this work optimal processes in linear systems were studied. It is demonstrated that such optimal processes in such systems do exist and equations are found for optimal directions and optimal trajectories. The problem of synthesis of such systems with one direction parameter was also considered.

Introduction

In the theory of automatic control, considerable importance is attached to the maximal increasing of speed in the acting of a large number of various controllers and monitoring systems, in other words, one strives towards the fastest, and as is common to say, the optimal realization of the process of control and monitoring. A number of other technical problems also leads to problems in which one extremizes time.

A fairly large collection of technical problems of this type constitutes the type of problem considered in the work (1). The representation of a point (vector) $\vec{x} = (x^1, \ldots, x^n)$ of an n-dimensional phase space, X has the following simultaneous equations of motion

$$\dot{x}^i = f^i(x^1, \ldots, x^n; u^1, \ldots, u^r) = f^i(\vec{x}, \vec{u}), \quad i = 1, \ldots, n. \qquad (1)$$

Here $(u^1, \ldots, u^r) = \vec{u}$ are the direction parameters. If the controlling law is given, i.e., we have r functions $u^1(t), \ldots u^r(t)$ belonging to some class of functions, then the system (1) for given initial conditions $x^i(t_0) = x^0_i$ determines uniquely the motion of the point $\vec{x}$ in the phase space. The class of functions from which one chooses the controlling functions $u^1(t), \ldots, u^r(t)$ depends upon the specific

details of the problem. A necessary condition is the stipulation that the vector $\vec{u}(t) = (u^1(t), \ldots, u^r(t)$ belonging to the r-dimensional space, should belong to some defined closed subspace of that space, for example, to an r-dimentional unit cube $|u^i(t)| \leqslant 1$, $i = 1, \ldots, r$. The physical meaning of this condition is obvious. Only some, or even no parameters may assume artitrarily large values, e.g., the controlling parameter may be the quantity of fuel fed into an engine, etc. Here, in the general mathmatical problem, the region in which the vector $\vec{u}(t)$ varies, does not have to be always bounded. In particular, it may coincide with the r-dimensional space.

Moreover, the control functions $u^1(t), \ldots, u^r(t)$ may belong to the class of piecewise continuous functions with a finite discontinuity. This corresponds to "inertialess" control, and the control parameters in the given case may instantaneously change value. However, in a number of technical problems, one has to take into account "inertia" of some control parameters, and hence, some of the functions $u^i(t)$ in this case shoudl be continuous and piecewise smooth with a finite derivative.

The class of vector functions $\vec{u}(t)$, from which one chooses the controls for the given problem, we shall call the <u>class of admissible</u> controls. This class shall be precisely defined below for the problems which we consider.

In addition to defining the class of admissible controls, the fundamental problem which here presents itself, is formulated in the following way.

In the phase space X there are two points $\vec{\xi}_0$, $\vec{\xi}_1$. One must choose an admissible control $\vec{u}(t) = (u^1(t), \ldots, u^r(t))$ in such a way that the phase point would travel on the trajectory of system (1) from point $\vec{\xi}_0$ to point $\vec{\xi}_1$ in a minimum time.

The required control $\vec{u}(t)$, if it exists, we shall call the optimal control and the corresponding trajectory -- the optimal trajectory.

The problem which is interesting in the theory of automatic contorl, has a slightly more specific character and may be stated in the following manner.

From an arbitrary, <u>a priori</u> determined, point belonging to the phase space X one should arrive at the origin of the coordinate system (1), with the aid of an admissible control.

Generally speaking, one may not arrive at the origin of the coordinate system from an arbitrary point belonging to the phase space X, by means of an admissible control. Let us denote by M the set of those points belonging to the space X from which one may arrive at the origin of the coordinate system by means of an optimal control. Then, on the set M we define a vector function

$$\vec{u}(\vec{x}) = (u^1(\vec{x}), \ldots, u^r(\vec{x})), \quad \vec{x} \in M, \tag{2}$$

whose values lie in the domain of admissible values of the control $\vec{u}$, and which satisfies the following condition: if one travels on the trajectories of the system

$$\dot{x}^i = f^i(\vec{x}, \vec{u}(\vec{x})), \quad \vec{x}(0) = \vec{\xi} \in M,$$

ıen the phase point $\vec{x}$, whose arbitrary given initial position is $\vec{\xi} \in M$ rrives at the origin of the coordinate system in minimum time.

The finding of the function $\vec{u}(x)$ is called the synthesis of the ptimal system. With the aid of this function, one builds the optimal mputing machine for the given controlling installation. The denoting the computer in the system of automatic control may be done schematically in the following manner (cf. (2)).

Assume that we have given a part of a system of automatic control consisting of the controlled element and a controlling factor. ıis part has r inputs and n outputs. The input quantities are the ntrol functions: $u^1(t), \ldots, u^r(t)$, the output quantities are the ntrolled phase coordinates $x^1(t), \ldots, x^n(t)$; the connection between ıe quantities u^i and x^j is given by system (1). The required pattern based on the fact that all phase coordinates x^i, as the process of ıe controlled element continues, should tend to zero.

The output quantities $(x^1, \ldots, x^n) = \vec{x}$ enter the computer, ıose output coincides with the input of the controlling element. ıus, one obtains a closed system of automatic control. If, as a sult of external factors, the phase point translates from the origin f the coordinate system to the point $\vec{x}$, where $\vec{x} = (x^1, \ldots, x^n)$, then t the input of the regulator we have from the computer the quantities $(\vec{x}) = (u^1(\vec{x}), \ldots, u^r(\vec{x}))$, which are defined by equation (2), which end the phase point back onto the origin, in a minimum of time.

In this work we consider the case of a linear regulator, i.e., e have the following system

$$\dot{x}^i = \sum_{\alpha=1}^{n} a^i_\alpha x^\alpha + b^i_1 u^1 + \ldots + b^i_r u^r = a^i_\alpha x^\alpha + b^i_\lambda u^\lambda + \ldots + b^i_r u^r,^* \quad i = 1, \ldots, n,$$

hich we write as a vector

$$\dot{\vec{x}} = A\vec{x} + \vec{b}_1 u^1 + \ldots + \vec{b}_r u^r. \tag{3}$$

ere $\vec{x}$ is a vector which is in an n-dimensional phase space X, A is a onstant linear transformation of this space, $\vec{b}_j$ are constant vectors n X. We define a class of admissible controls to be the class of iecewise continuous functions whose modulus is at most unity, i.e., $|u^j| \leq 1$. Generally speaking, unless the last restriction is made n the control function, the problem has no solution for (3), and one ay move from a point $\vec{\xi}_0$ to another point $\vec{\xi}_1$ in an arbitrarily short nterval of time; however, when the transformation time tends to zero, he corresponding controls have their moduli increased to arbitrarily arge quantities.

The existence theorem which is proved in para 3 states this:

If one may reach from point $\vec{\xi}_0$ to point $\vec{\xi}_1$ by some admissible ontrol, then one may go from the first point to the latter following n optimal control.

* Further in the text we shall adopt tensor notation.

This control is a relay i.e., the control functions have values $u^j(t) = \pm 1$ and have a finite number of discontinuities (jumps).

In para 1 we have obtained the equations for optimal controls and trajectories.

In para 2 we study the problem of optimizing the system (3) for $r = 1$, on the basis of results from para 1 and para 3 (the existence theorems).

The existence theorem proved in para 3 is not only of mathematical importance. We shall prove that the optimal control u^j, $j = 1, \ldots, r$ for equation (3), one may always find in the class of relay controls. Therefore, one does not exclude the case when there exists a sequence of relay controls u^j_k, $k = 1, 2, \ldots$ which transform the phase point along the trajectory of equation (3) from position ξ_0 to the point ξ_1, after a lapse of time t_k, where

$$t_1 > t_2 > \ldots > t_k > \ldots > T,$$

with the proviso that there does not exist an admissible control u^j, $j = 1, \ldots, r$, such that it transforms the phase point from point ξ_0 onto ξ_1 in time T. In this case it is not permissible to choose a subsequence of the sequence of controls u^j_k such that it should converge to some permissible control u^j as this control would transform the phase point from position ξ_0 onto point ξ_1 in time T. Consequently, the totality of points of jumps of the control u^j_k increases without limit together with the number k, and the relay control u^j_k at certain times or during the whole time of travel from ξ_0 to ξ_1, begins to oscillate at a high frequency.

Thus, we are unable to obtain an optimal transformation from ξ_0 to ξ_1, however the trajectory shall be the nearer to the optimal trajectory, the faster shall oscillate the controls u^j_k at the respective transformation time intervals. Such regions are well known in the theory of automatic control and are called sliding regions.

Therefore, the existence theorem for optimal processes for system (3) is equivalent to the following assertion:

In linear systems the optimal region may not be sliding for any time interval.

One may consider a more general problem and assume that some part of the control functions $u^i(t)$ may be chosen in the class of continuous piecewise smooth functions with a bounded (with respect to modulus) derivative, i.e., to assume that the respective u^i exhibit "inertia" while another part considered to be "inertialess" is obtained from previous conditions. In some cases, one of these parts, only, may exist. The results, relevant to this general case, shall be published separately.

The methods of the present work are a natural extension of the method published in the article [1]. The present work was completed in the seminar of L. S. Pontryagin on the mathematical theory of oscillations and automatic control.

I express my deep gratitude for his attention and great

assistance, which he afforded me during the completion of the present work.

Para 1. Equations for Optimal Controls and Trajectories for the Case of One Control Parameter

1. Statement of the problem. Notations. Let there be given one differential linear equation with one control (scalar) parameter u:

$$\dot{\vec{x}} = A\vec{x} + \vec{b}u. \tag{4}$$

Here $\vec{x}$ is the representative point (vector) in an n-dimensional phase space X, $\vec{b}$ is a constant vector in space X, A is a linear transformation of space X, which is independent of time. The control function u belongs to the class of piecewise continuous functions (with a finite number of points of discontinuity on every closed interval of time), such that the modulus is at most unity. Such functions we shall call admissible controls.

Statement of general problem. There exist two points $\vec{\xi}_0$ and $\vec{\xi}_1$ in the phase space. One seeks to find such an admissible control function $u = u(t)$, such that the representative point $\vec{x}(t)$ travelling on the trajectory defined by the equation (4) should travel from point $\vec{\xi}_0$ to point $\vec{\xi}_1$ in a minimum of time.

Such a control function, if it exists, we shall call the optimal control and the corresponding trajectory, the optimal trajectory. We shall also say that the phase point executes an optimal passage from the point $\vec{\xi}_0$ to point $\vec{\xi}_1$ along the trajectory defined by the equation (4) in an optimum of time.

Let us introduce some new notations which are necessary for further presentation. The vectors, which belong to the phase space X, we shall call contravariant, those which are in the dual space of X we shall call covariant.

Let $\vec{\varphi}_1(t), \ldots, \vec{\varphi}_n(t)$ be contravariant vector functions, which represent a fundamental system of solutions of the equation $\dot{\vec{x}} = Ax$. Let $\vec{\psi}^1(t), \ldots, \vec{\psi}^n(t)$ be the covariant vector functions, dual, for equal indices, to the functions $\vec{\varphi}_i(t)$:

$$\vec{\varphi}_i(t)\vec{\psi}^j(t) = \delta_i^j.$$

We have:

$$\dot{\vec{\varphi}}_i = -A\vec{\varphi}_i, \quad i = 1, \ldots, n, \tag{5}$$
$$\dot{\vec{\psi}}^j = -A'\vec{\psi}^j, \quad j = 1, \ldots, n, \tag{6}$$

where A' is a linear transformation which is a dual of A.

In order to prove (6), in other words, that the vector functions $\vec{\psi}^i(t)$ satisfy the dual (of (6)) equation (5), we differentiate the relation $\vec{\varphi}_\alpha(t)\cdot\vec{\psi}^i(t) = \delta_\alpha^i$. From it we have:

$$(\vec{\varphi}_\alpha(t)\cdot\vec{\psi}^i(t))' = \dot{\vec{\varphi}}_\alpha(t)\cdot\vec{\psi}^i(t) + \vec{\varphi}_\alpha(t)\cdot\dot{\vec{\psi}}^i(t) =$$
$$= A\vec{\varphi}_\alpha(t)\vec{\psi}^i(t) + \vec{\varphi}_\alpha(t)\dot{\vec{\psi}}^i(t) = \vec{\varphi}_\alpha(t)\cdot A'\vec{\psi}^i(t) + \vec{\varphi}_\alpha(t)\cdot\dot{\vec{\psi}}^i(t) =$$
$$= \vec{\varphi}_\alpha(t)\cdot(A'\vec{\psi}^i(t) + \dot{\vec{\psi}}^i(t)) \equiv 0.$$

As this equation is satisfied for an arbitrary $\alpha = 1, \ldots, n$, the following identities are true:

$$A'\vec{\psi}^i(t) + \dot{\vec{\psi}}^i(t) \equiv 0, \quad i = 1, \ldots, n.$$

We denote the scalar product of the vector function $\vec{\psi}^i(t)$ with the vector $\vec{b}$, by
$$h^i(t) = \vec{\psi}^i(t)\cdot\vec{b}, \quad i = 1, \ldots, n.$$

The solution of equation (4) with the initial condition $\vec{x}(0) = \vec{\xi}(0) = \vec{\varphi}_\alpha(0)\,\xi^\alpha$ is written as follows:

$$\vec{x}(t) = \vec{\varphi}_\alpha(t)\left(\xi^\alpha + \int_0^t h^\alpha(\tau)\,u(\tau)\,d\tau\right) = \vec{\varphi}_\alpha(t)\left(\xi^\alpha + \int_0^t \vec{\psi}^\alpha(\tau)\cdot\vec{b}\,u(\tau)\,d\tau\right). \tag{7}$$

As is known, every function $h^i(t)$, $i = 1, \ldots, n$ is a solution of an nth order equation, namely

$$|A' + pE|\,h(t) = 0, \tag{8}$$

where p is the differentiation operator, E is the identity transformation, $|A' + pE|$ is the characteristic transformation polynomial for A'.

2. <u>The Condition of Nondegeneracy</u>. The equation (4) is nondegenerate if the vector $\vec{b}$ does not lie in some invariant subspace, whose dimension is at most $n - 1$, with respect to the transformation A. Otherwise, the equation (4) is degenerate.

Now we shall show that if equation (4) is degenerate, then the time it takes to go from $\vec{\xi}_0$ to $\vec{\xi}_1$ is independent of the choice of the control function u(t), or the problem reduces itself to a similar problem for an equation of smaller order.

Let us assume that $\vec{b} \in Y$, where Y is an invariant subspace of the transformation A, where dim (y) = n - 1. Let us express the phase space X as the direct sum of Y and Z, X = Y + Z, and every vector $\vec{x} \in X$, as a sum $\vec{x} = \vec{y} + \vec{z}$, where $\vec{y} \in Y$ and $\vec{z} \in Z$. The equation (4) becomes:

$$\dot{\vec{x}} = \dot{\vec{y}} + \dot{\vec{z}} = A\vec{y} + \vec{b}u + A\vec{z},$$

where $A\vec{y} \in Y$, $\vec{b}u \in Y$. The projection operator onto the subspace Y, parallel to subspace Z, is denoted by Pr_1, the operator onto Z, parallel to Y, we denote by Pr_2. We have:

$$Pr_1\dot{\vec{x}} = \dot{\vec{y}} = A\vec{y} + \vec{b}u + Pr_1A\vec{z}, \quad Pr_2\dot{\vec{x}} = \dot{\vec{z}} = Pr_2A\vec{z} = B\vec{z},$$

where B is some linear transformation in Z.

Assume that one wants to get to $\vec{\xi}_1 = (\vec{y}_1, \vec{z}_1)$ by an admissible function (not necessarily optimal), starting from the point $\vec{\xi}_0 = (\vec{y}_0, \vec{z}_0)$. The corresponding trajectory of equation (4) we denote by $\vec{x}(t) = (\vec{y}(t), \vec{z}(t))$, where

$$(\vec{y}(0), \vec{z}(0)) = (\vec{y}_0, \vec{z}_0), (\vec{y}(t_1), \vec{z}(t_1)) = (\vec{y}_1, \vec{z}_1).$$

The vector function $\vec{z}(t)$ is called a solution of the equation $\dot{\vec{z}} = B\vec{z}$, and is independent of the choice of control function u(t).

If $B\vec{z}_o = 0$, then it is necessary for the solvability of the problem that the equality $\vec{z}_o = \vec{z}_1$ should be satisfied, as independently from the choice of the control function, we shall have the identity $\vec{z}(t) \equiv \vec{z}_o$. In this case the problem of optimal motion from point $(\vec{y}_o, \vec{z}_o)$ in the phase space X to point $(\vec{y}_1, \vec{z}_1)$, along the trajectory of equation (4), reduces to a problem of optimal motion from point $\vec{y}_o$ in phase space Y, to point $\vec{y}_1$, on the trajectory of the equation

$$\dot{\vec{y}} = A\vec{y} + \vec{b}u + Pr_1 A\vec{z}_0 = A\vec{y} + \vec{b}u + \vec{a},$$

where $Pr_1 A\vec{z}_o = \vec{a}$ is some constant vector in the space Y. This equation may be taken as is, and as equation of type (4), it does not contain the vector $\vec{a}$. However if the transformation A, which we consider in the invariant subspace Y, is not degenerate. then $\vec{a} = A\vec{a}_1$, and we have

$$\dot{\vec{y}} = (\vec{y} + \vec{a}_1)' = A(\vec{y} + \vec{a}_1) + \vec{b}u.$$

If $B\vec{z}_o \neq 0$ and there exists a trajectory $\vec{z}(t)$ of the equation $\dot{\vec{z}} = B\vec{z}$, which connects the points $\vec{z}_o$ and $\vec{z}_1$, then the time it takes for point $\vec{z}_o$ to arrive at $\vec{z}_1$ and the time it takes the phase point $\vec{x}$ to move from $(\vec{z}_o, \vec{y}_o)$ to $(\vec{z}_1, \vec{y}_1)$ is independent of the choice of the control function u(t), if, of course, the latter case of motion is usually possible as it is necessary to move from point $\vec{y}_o$ to point $\vec{y}_1$ also.

Further, we shall consider that equation (4) is nondegenerate. From the nondegeneracy of equation (4) it follows directly the linear independence of the v ctors $\vec{b} = A^o\vec{b}, \ldots, A^{n-1}\vec{b}$. Hence, it is easy to deduce that the functions $h^i(t) = \vec{\psi}^i(t) \cdot \vec{b}$, $i = 1, \ldots, n$ (cf. 1) are linearly independent. Indeed, let

$$c_\alpha h^\alpha(t) = c_\alpha \vec{\psi}^\alpha(t) \cdot \vec{b} \equiv 0,$$

where c_α, $\alpha = 1, \ldots, n$ are constants, then

$$c_\alpha \dot{\vec{\psi}}^\alpha \vec{b} \equiv \ldots = c_\alpha \overset{(n-1)}{\vec{\psi}^\alpha} \vec{b} \equiv 0.$$

From formula (6) and the above equations it follows that:

$$c_\alpha \vec{\psi}^\alpha \cdot \vec{b} \equiv c_\alpha \vec{\psi}^\alpha \cdot A\vec{b} \equiv \ldots \equiv c_\alpha \vec{\psi}^\alpha \cdot A^{n-1}\vec{b} \equiv 0.$$

As the vectors $\vec{b}, A\vec{b}, \ldots, A^{n-1}\vec{b}$ are independent, then $c_\alpha \vec{\psi}^\alpha(t) \equiv 0$, and as the vectors $\vec{\psi}^\alpha(t)$, $\alpha = 1, \ldots, n$ represent a fundamental system of solutions of equation (6), then $c_1 = \ldots = c_n = 0$.

From the linear dependence of functions $h^1(t), \ldots, h^n(t)$ it follows that they represent a fundamental system of solutions of an nth order equation (8).

3. Necessary Conditions for Optimality. Let $\vec{\xi}_o$ be some defined point of phase space X. Let us denote by $\Omega(t)$, $t \geq 0$, the set of all points of the space X such that one may reach them after a time t moving along the trajectory of equation (4), by means of an arbitrary admissible control, for an initial condition $\vec{x}(0) = \vec{\xi}_o$.

The set $\Omega(t)$ is convex for an arbitrary $t \geqslant 0$. As a matter of fact if u_1 and u_2 are two admissible controls and their respective trajectories are $\vec{x}_1(t)$ and $\vec{x}_2(t)$, of the equation (4) and satisfy the initial conditions

$$x_1(0) = x_2(0) = \xi_0;$$

then $\vec{x}_1(t) \in \Omega(t)$, $\vec{x}_2(t) \in \Omega(t)$ for an arbitrary $t \geqslant 0$. An arbitrary point of an arc connecting the points $\vec{x}_1(t)$, $\vec{x}_2(t)$ may be written as

$$\lambda \vec{x}_1(t) + \mu \vec{x}_2(t),$$

where $\lambda + \mu = 1$, $\lambda \geqslant 0$, $\mu \geqslant 0$. One may reach this point from a point $\vec{\xi}_0$ in time t, moving along a trajectory of the equation (4) by means of a control $\lambda u_1 + \mu u_2$, such that the corresponding trajectory is (cf formula (7)):

$$\vec{\varphi}_\alpha(t)\left(\xi_0^\alpha + \int_0^t h^\alpha(\lambda u_1 + \mu u_2)\,d\tau\right) = \lambda\vec{\varphi}_\alpha(t)\left(\xi_0^\alpha + \int_0^t h^\alpha u_1\,d\tau\right) +$$

$$+ \mu\vec{\varphi}_\alpha(t)\left(\xi_0^\alpha + \int_0^t h^\alpha u_2\,d\tau\right) = \lambda\vec{x}_1(t) + \mu\vec{x}_2(t),$$

where $\varphi_\alpha(0)\,\xi_0^\alpha = \vec{\xi}_0$ is the starting point of the trajectory; the direction $\lambda u_1 + \mu u_2$ is admissible, provided that

$$|\lambda u_1 + \mu u_2| \leqslant \lambda|u_1| + \mu|u_2| \leqslant \lambda + \mu = 1.$$

Let now u(t) be the optimal control, $\vec{x}(t)$ the corresponding optimal trajectory of equation (4), which connects the given point $\vec{\xi}_0$ with some point $\vec{\xi}_1$ which lies in X. Moreover, let

$$\vec{x}(0) = \vec{\xi}_0, \quad \vec{x}(T) = \vec{\xi}_1.$$

The control u(t) and the trajectory x(t) are defined to be optimal on an arbitrary interval $0 \leqslant t \leqslant T_1$, where $T_1 \leqslant T$. Actually, if one may arrive at point $\vec{x}(T_1)$ from the point $\vec{\xi}_0$ in time $T_1 - \varepsilon, \varepsilon > 0$ by means of an admissible control v(t), where $0 \leqslant t \leqslant T_1 - \varepsilon$, then one may reach the point $\vec{x}(T)$ in time $T - \varepsilon$, by means of an admissible direction w(t), $0 \leqslant t \leqslant T - \varepsilon$, where $w(t) = v(t)$ for $0 \leqslant t \leqslant T_1 - \varepsilon$, and $w(t) = u(t + \varepsilon)$ for

$$T_1 - \varepsilon \leqslant t \leqslant T - \varepsilon,$$

which contradicts the assumption of optimality of the trajectory $\vec{x}(t)$ on the interval $0 \leqslant t \leqslant T$.

We shall show now that we may construct a hyperplane P which contains the point $\vec{\xi}_1 = \vec{x}(T)$ and which is a base for the convex set $\Omega(T)$, i.e., that $\vec{\xi}_1$ is in the boundary of the set $\Omega(T)$.

It is easy to see that for an arbitrary $t \leqslant T$ the point $\vec{x}(t)$ is a boundary point of the set $\Omega(t)$ and because of that, one may have through it a supporting hyperplane to $\Omega(t)$. If this is not true for even one point $\vec{x}(t)$, where $t \leqslant T$, then we shall consider a spherical neighborhood about $\vec{x}(t)$, such that it is properly contained in the set $\Omega(t)$ and shall define in the ball any point $\vec{x}(t + \varepsilon)$ of the trajectory, where $\varepsilon \geqslant 0$ (such a ball may be found as soon as the defined point $\vec{x}(t)$ is an interior point of the set $\Omega(t)$. As this ball is

properly contained in the set $\Omega(t)$, then one may proceed from point $\vec{\xi}_0$ to point $x(t + \varepsilon)$ via an admissible control in time $t < t + \varepsilon$, which contradicts the optimality of the trajectory $\vec{x}(t)$.

Therefore, one may select a sequence of times $t_k \to T$ and a sequence of P_k hyperplanes which pass correspondingly through $\vec{x}(t_k)$ and are supporting hyperplanes for $\Omega(t_k)$. Moreover, assume that the sequence of hyperplanes P_k converges to some hyperplane P, which passes through the point $\vec{\xi}_1 = \vec{x}(T)$. The hyperplane P is the supporting hyperplane for $\Omega(T)$.

Let us denote $\vec{\chi}_k$ the covariant vector, which is orthogonal to the hyperplane P_k, away from the set $\Omega(t_k)$, i.e., if $\vec{y} \in \Omega(t_k)$, then $(\vec{y} - \vec{x}(t_k))\vec{\chi}_k \leq 0$. One may assume that $\vec{\chi}_k \to \vec{\chi}$, where $\vec{\chi}$ is orthogonal to P. Further, we shall denote an arbitrary admissible control $u(t) + \delta(u(t))$, where $u(t)$ is the investigated optimal control, $\delta u(t)$ is an arbitrary admissible perturbation of the direction $u(t)$; by $\vec{x}(t) + \delta\vec{x}(t)$ we shall denote the corresponding perturbed trajectory with the initial condition $\delta\vec{x}(0) = 0$.

For an arbitrary $t < T$ the point $\vec{x}(t_k) + \delta\vec{x}(t_k) \in \Omega(t_k)$ and $\vec{x}(t_k) \in P_k$ hence $\delta\vec{x}(t_k)\vec{\chi}_k \leq 0$. Proceeding to the limit, for $k \to \infty$, we obtain for an arbitrary admissible perturbation $\delta\vec{x}$, that $\delta\vec{x}(T) \cdot \vec{\chi} \leq 0$, i.e., that the hyperplane P passing through the point $\vec{x}(T)$ is the supporting plane for $\Omega(T)$, and hence, the point itself $(\vec{x}(T))$ lies on the boundary of the set $\Omega(T)$.

We shall show that the supporting hyperplane P may always be found for a point $\vec{x}(T)$ and a vector $\vec{\chi}$, which is orthogonal to it, such that in addition to the inequality $\delta\vec{x}(T) \cdot \vec{\chi} \leq 0$, also the inequality

$$\dot{\vec{x}}(T) \cdot \vec{\chi} = (A\vec{x}(T) + \vec{b}u(T)) \cdot \vec{\chi} \geq 0.$$

would be satisfied.

For this, we shall prove that the beam which contains the phase velocity $\dot{\vec{x}}(t)$ and which emanates from point $\vec{x}(t)$ for $0 \leq t \leq T$, contains only the boundary points of $\Omega(t)$.

This theorem is easy to prove for $t < T$, thus, passing to the limit to prove it for $t = T$.

Let us assume that the beam q_t, which contains the vector $\dot{\vec{x}}(t)$, where $t < T$, and emanating from the point $\vec{x}(t)$, contains an interior point $\vec{y}$ of the set $\Omega(t)$. As $\Omega(t)$ is a convex set, then the whole interval of the beam q_t between the points $\vec{y}$ and $\vec{x}(t)$ consists entirely of interior points of the set $\Omega(t)$. Furthermore, the phase velocity vector $\dot{\vec{x}}(t)$ is different from zero. It follows that one may find such a time t', $t < t' < T$, such that $\vec{x}(t') \in \Omega(t)$ which is impossible, as the $t < t'$ and the trajectory is an optimal one.

From it it follows that q_t does not contain the interior points of the set $\Omega(t)$. This implies the possibility of two cases: either the straight line Q_t, which contains the beam q_t, contains interior points of $\Omega(t)$ or not. In the first case it is evident that the covariant vector $\vec{\chi}_t$ which is orthogonal to an arbitrarily chosen supporting hyperplane P_t, relative to the set $\Omega(t)$ at point $\vec{x}(t)$ and a direction away from the set $\Omega(t)$ satisfies the inequality

$$\dot{\vec{x}}(t) \cdot \vec{\chi}_t \geq 0.$$

In the second case the supporting hyperplane P_t may be chosen in such a way that it contains a straight line Q_t, thus, in either case, the inequality

$$\vec{x}(t)\cdot\chi_t \geqslant 0$$

is satisfied.

Take a sequence $t_k \longrightarrow T$, and perform a similar construction for every point $\vec{x}(t_k)$, when we shall obtain in the limit an inequality

$$\vec{x}(T)\cdot\chi \geqslant 0.$$

We shall now state in its entirety the necessary condition.

Let u(t) be the optimal control and let $\vec{x}(t)$ be the corresponding optimal trajectory of the equation (4), which connects the point $\vec{\xi}_0$ to the point $\vec{\xi}_1$, both of which belong to the phase space X:

$$\vec{x}(0)=\vec{\xi}_0,\quad \vec{x}(T)=\vec{\xi}_1,\quad 0\leqslant t\leqslant T.$$

Then, we are able to pass through $\vec{\xi}_1 = \vec{x}(T)$ a hyperplane P which is supporting relative to $\Omega(T)$, in such a manner that, if we denote by $\vec{\chi}$ the covariant vector which is orthogonal to P and is directed away from the set $\Omega(T)$, then the following inequalities

$$\delta\vec{x}(T)\cdot\vec{\chi}\leqslant 0,\quad \dot{\vec{x}}(T)\cdot\vec{\chi}=(A\vec{x}(T)+\vec{b}u(T)\vec{\chi})\geqslant 0$$

are satisfied for arbitrary admissible perturbations $\delta\vec{x}(t)$ of the optimal trajectory $\vec{x}(t)$.

4. <u>Optimal Controls and Optimal Trajectories Equations</u>.

By means of the stated necessary condition, it is easy to obtain equations for the optimal control u(t) and the optimal trajectory $\vec{x}(t)$, which connects the point $\vec{\xi}_0 = \vec{x}(0)$ with the point $\vec{\xi}_1 = \vec{x}(T)$, $0 \leqslant t \leqslant T$. To this end, we shall use the inequality $\delta\vec{x}(T)^1\cdot\vec{\chi}\leqslant 0$. It is seen, that the perturbation $\delta\vec{x}(t)$ satisfies the differential equation

$$\delta\dot{\vec{x}} = A\delta\vec{x}+\vec{b}\,\delta u$$

with the initial conditions $\delta\vec{x}(0) = 0$. Thus, on the basis of formula (7), we have

$$\delta\vec{x}(T)\cdot\vec{\chi}=\vec{\chi}\cdot\vec{\varphi}_\alpha(T)\int_0^T\vec{\psi}^\alpha\cdot\vec{b}\,\delta u\,d\tau=$$

$$=\int_0^T c_\alpha\vec{\psi}^\alpha\cdot\vec{b}\,\delta u\,d\tau=\int_0^T c_\alpha h^\alpha\,\delta u\,d\tau\leqslant 0,$$

where $c_\alpha = \vec{\chi}\cdot\vec{\varphi}_\alpha(T)$, $\alpha = 1, \ldots, n$, do not become zero simultaneously and

$$\vec{h}^\alpha(t)=\vec{\psi}^\alpha(t)\cdot\vec{b}\qquad \text{(cf. no: 1)}$$

The function $c_\alpha h^\alpha(t)$ does not vanish identically, as not all the c_α are zero, and the functions $h^\alpha(t)$ represent, as a consequence of the nondegeneracy of equation (4), a fundamental system of solutions of an equation of the nth order (8) (cf. no 1-2). The inequality

$$\int_0^T c_\alpha h^\alpha \delta u d\tau \leqslant 0$$

is true for every admissible perturbation of the optimal control. Thence it follows that for positive values of the functions $c_\alpha h^\alpha(t)$, the perturbation $\delta u(t)$ may only assume values which are not greater than zero, and for negative values of the functions $c_\alpha h^\alpha(t)$ only non-negative values. As the inequality $|u(t) + \delta u(t)| \leq 1$, is a sufficient condition for the admissibility of the perturbed direction then we obtain the following equation for the optimal control $u(t)$:

$$u(t) = \operatorname{sign} c_\alpha h^\alpha(t) = \operatorname{sign} c_\alpha \vec{\psi}^\alpha(t) \cdot \vec{b} = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}, \quad 0 \leqslant t \leqslant T. \tag{9}$$

The function which assumes only the values ± 1, we shall call a relay function from now on.

The covariant vector function $c_\alpha \vec{\psi}^\alpha(t) = \vec{\psi}(t)$ satisfies the equation (6):

$$\dot{\vec{\psi}} = -A'\vec{\psi} \tag{10}$$

and possesses a simple geometrical meaning: the vector $\vec{\psi}(t)$ for every t, $0 \leq t \leq T$ defines uniquely a hyperplane P_t, which is orthogonal to it, such that it passes through the point $\vec{x}(t)$ and is supporting relative to the set $\Omega(t)$, and is directed away from the set $\Omega(t)$. For an arbitrary admissible perturbation $\delta(u(t))$ we have

$$\delta\vec{x}(t)\cdot\vec{\psi}(t) = \vec{\psi}(t)\cdot\vec{\varphi}_\alpha(t)\int_0^t \vec{\psi}^\alpha\cdot\vec{b}\,\delta u\, d\tau = c_\beta\vec{\psi}^\beta(t)\cdot\vec{\varphi}_\alpha(t)\int_0^t \vec{\psi}^\alpha\cdot\vec{b}\,\delta u\, d\tau;$$

As the systems $\{\vec{\psi}^\alpha(t)\}$ and $\{\vec{\varphi}_\alpha(t)\}$, $\alpha = 1, \ldots, n$, (cf. no 1) are dual to one another, we have

$$\vec{\psi}^\beta(t)\cdot\vec{\varphi}_\alpha(t) = \delta^\beta_\alpha,$$

i.e.,

$$\delta\vec{x}(t)\cdot\vec{\psi}(t) = \int_0^t c_\alpha\vec{\psi}^\alpha\cdot\vec{b}\,\delta u d\tau = \int_0^t \vec{\psi}\cdot\vec{b}\,\delta u d\tau.$$

On the other hand, from equation (9) and the inequality $|u(t) + \delta u(t)| \leq 1$ it follows that $\operatorname{sign}\delta u(t) = -\operatorname{sign}\vec{\psi}(t)\cdot\vec{b}$ and, hence, $\delta\vec{x}(t)\cdot\vec{\psi}(t) \leq 0$. In particular $P_T = P$ and $\vec{\psi}(T) = \vec{\chi}$, as

$$\vec{\psi}(T) = c_\alpha\vec{\psi}^\alpha(T) = (\vec{\chi}\cdot\vec{\varphi}_\alpha(T))\vec{\psi}^\alpha(T) = \vec{\chi}$$

(as a consequence of the duality of the system $\{\vec{\varphi}_\alpha(T)\}$ and $\{\vec{\psi}^\alpha(T)\}$, $\alpha = 1, \ldots, n$).

At the end point $\vec{\xi}_1 = \vec{x}(T)$, the inequality $\vec{\psi}(T)\cdot\vec{x}(T) \geqslant 0$ holds, as may be seen from no 3.

Now, we shall prove that everywhere along the optimal trajectory $\vec{x}(t)$, the scalar product $\vec{\psi}(t)\cdot\vec{x}(t)$ is constant, hence,

$$\vec{\psi}(t)\cdot\vec{x}(t) = \text{const} \geqslant 0.$$

The scalar product

$$\vec{\psi}(t)\cdot\dot{\vec{x}}(t) = \vec{\psi}(t)\cdot(A\vec{x}(t) + \vec{b}u(t)) \tag{11}$$

is continuous, so from (9) we see that u(t) may jump only when $\vec{\Psi}(t) \cdot \vec{b} = 0$. Thus, the constancy of the scalar product shall follow from the fact that the first derivative of the scalar product with respect to time is identically zero. From equations (4), (9), and (10), we have:

$$\begin{aligned}(\vec{\psi}(t)\cdot(A\vec{x}(t)+\vec{b}u(t)))' &= \dot{\vec{\psi}}(t)\cdot A\vec{x}(t)+\vec{\psi}(t)\cdot A\dot{\vec{x}}(t)+\\ &+\vec{\psi}(t)\cdot\vec{b}\dot{u}(t) = -A'\vec{\psi}(t)\cdot A\vec{x}(t)+\vec{\psi}(t)\cdot A(A\vec{x}(t)+\vec{b}u(t))+\\ &+(-A'\vec{\psi}(t)\cdot\vec{b}u(t)) = -\vec{\psi}(t)\cdot A^2\vec{x}(t)+\vec{\psi}(t)\cdot A^2\vec{x}(t)+\\ &+\vec{\psi}(t)\cdot A\vec{b}u(t)-\vec{\psi}(t)\cdot A\vec{b}u(t)\equiv 0.\end{aligned}$$

Having collected the conditions stated in equations (4), (9), (10), (11), we obtain the following method for the determination of optimal controls and optimal trajectories emanating out of the given point $\vec{\xi}_0$, which belongs to the phase space X.

All optimal controls u(t) and their respective optimal trajectories $\vec{x}(t)$, which emanate from the point $\vec{\xi}_0$ at t = 0, are contained in a set of all controls and their respective trajectories, which are obtained upon solving the system of equations as stated below:

$$\left.\begin{aligned} &\dot{\vec{x}} = A\vec{x}+\vec{b}u,\ \vec{x}(0)=\vec{\xi}_0,\\ &\dot{\vec{\psi}} = -A'\vec{\psi},\\ &u(t) = \operatorname{sign}\vec{\psi}(t)\cdot\vec{b},\\ &\vec{\psi}(0)\cdot\dot{\vec{x}}(0) = \vec{\psi}(0)\cdot(A\vec{x}(0)+\vec{b}u(0)) \geqslant 0.\end{aligned}\right\} \qquad (12)$$

Since we are not interested in the vector function $\vec{\Psi}(t)$ itself, but merely in the control function u(t) = sign $\vec{\Psi}(t) \cdot \vec{b}$, we henceforth regard the vector $\vec{\Psi}(0) = c_\alpha \vec{\Psi}^\alpha(0)$ as normalized, i.e., define the equation

$$\sum_{\alpha=1}^{n} c_\alpha^2 = \|\vec{\psi}(0)\| = 1.$$

to hold, for not all c_α equal to zero.

Provided that inequality $\vec{\Psi}(0) \cdot \dot{\vec{x}}(0) \geqslant 0$ is satisfied, we let the initial value of $\vec{\Psi}(0)$ assume any arbitrary value and thus we obtain a set of controls and trajectories which emanate from point $\vec{\xi}_0$, which we shall call extremal controls and extremal trajectories, amongst which shall be all optimal controls and optimal trajectories which emanate from point $\vec{\xi}_0$.

If we know beforehand, that by choosing an optimal control, and consequently, a unique extremal trajectory, we may proceed from a point $\vec{\xi}_0$ to some other point $\vec{\xi}_1$, both of which lie in the phase space X, along the aforementioned trajectory, then the corresponding extremal control is an optimal control. This happens with the synthesis of an optimal system, which shall be discussed in the next paragraph.

The system of equations (12) expresses exactly the maximum principle, which has been formulated in work (1). The essence of this principle may be stated as follows. The first two equations of system (12) may be written in the sense of Hamilton, using the hamiltonian $H(\vec{x}, \vec{\Psi}, u) = \vec{\Psi}\cdot(A\vec{x} + \vec{b}u) = \vec{\Psi}\dot{\vec{x}}$. These become:

$$\dot{\vec{x}} = \frac{\partial H}{\partial \psi} = \frac{\partial}{\partial \psi}(\vec{\psi}\cdot(A\vec{x} + \vec{b}u)) = A\vec{x} + \vec{b}u,$$
$$\dot{\vec{\psi}} = -\frac{\partial H}{\partial x} = -\frac{\partial}{\partial x}(\vec{\psi}\cdot(A\vec{x} + \vec{b}u)) = -\frac{\partial}{\partial x}A'\vec{\psi}\vec{x} = -A'\vec{\psi}.$$

The equation $u(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}$ is equivalent to the following condition: for any t, the control u(t) assumes an admissible value, for which the function $H = \vec{\psi} \cdot (A\vec{x} + \vec{b}u)$, considered to be a function of the independent variable u, for constant $\vec{x}$ and $\vec{\psi}$, assumes a maximum value. This condition is called the Maximum Principle.

In conclusion, it follows from formula (11) that the hamiltonian function along the whole extremal trajectory $H(\vec{x}(t), \vec{\psi}(t), u(t)) = \text{const} \geqslant 0$.

Para 2. The Synthesis of Optimal Linear Systems With One Control Parameter

1. Formulation of the problem. In the theory of automatic control, one is interested in the optimal traverse from an arbitrary initial position to the origin of a coordinate system, along a trajectory defined by equation (4) (cf. the introduction).

In this paragraph, we shall assume that the existence theorem for the optimal processes for equation (4), are proved (cf. para 3, no 2).

In No 2 of this paragraph, we show that only one extremal trajectory, expressed by equation (4), leads from an arbitrary point of the phase space to the origin of the coordinate system, such that it is also the optimal trajectory (cf. para 1, no 4, para 3, no 2). Therefore, it is not necessary to distinguish between extremal trajectories which lead to the origin, with their respective extremal controls, and optimal trajectories and controls.

In No 3 we shall consider a set of all those points of the phase space from which one may arrive at the origin of the coordinate system along an admissible, and therefore, an optimal, control. We shall denote this set by M. This set proves to be a convex domain in the phase space X. If the characteristic values of the transformation A are all distinct, the set M is identical with the space X.

At the basis of the theorem of uniqueness of the extremal trajectory which leads to the origin of coordinates, lies the fact that along such an extremal trajectory there exists an extremal control u(t) which is defined uniquely to be the function of a point lying on the trajectory, with the exclusion only of those points of the trajectory which correspond to the times at which the control jumps. The number of such times is a finite one in a finite time interval, as

$$u(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}$$

(cf. (12)), and the function $\vec{\Psi}(t) \cdot \vec{b}$ does not vanish identically and is a solution of an nth order linear equation (8) with constant coefficients. It is immaterial which values we shall ascribe to the control u(t) at the time of the jump, let us arbitrarily state that at such time u = 0.

It follows that the set M may be uniquely decomposed into a direct sum of three sets: $M = M_+ \cup M_- \cup M_0$ according to the following criterion. If a point $\vec{x}(t)$ which lies on the extremal trajectory belongs to M_+ then u(t) = 1, if $\vec{x}(t) \in M_-$, then u(t) = -1, finally, if $\vec{x}(t)$ belongs to M_0, then at that time u(t) jumps.

The dimensionality of the set M_0, therefore, is at most n - 1.

This enables us to obtain a monovalent function $u(\vec{x})$ which is defined on the set M by the following conditions:

$$u(\vec{x}) = 1 \Big| \text{ for } \vec{x} \in M_+, \quad u(\vec{x}) = -1 \Big| \text{ for } \vec{x} \in M_-, \quad u(\vec{x}) = 0 \Big| \text{ for } \vec{x} \in M_0 \Big|$$

which possesses the property such that, if it moves along the trajectory expressed by

$$\dot{\vec{x}} = A\vec{x} + \vec{b}u(\vec{x}), \quad \vec{x}(0) \in M. \Big| \tag{13}$$

then we shall arrive at the origin of coordinates from $\vec{x}(0)$ along the optimal trajectory defined by equation (4) after an optimal lapse of time.

The process of finding of such function $u(\vec{x})$ is the synthesis of the optimal system described by equation (4) (cf. introduction).

If the function $v(\vec{x})$ satisfies the conditions

$$v(\vec{x}) > 0 \Big| \text{ for } \vec{x} \in M_+, \quad v(\vec{x}) < 0 \Big| \text{ for } \vec{x} \in M_-, \quad v(\vec{x}) = 0 \Big| \text{ for } \vec{x} \in M_0, \Big|$$

then $u(\vec{x})$ = sign $v(\vec{x})$.

In order to find the function $v(\vec{x})$, it is very convenient to use the structure of the "set of jump points" of the function $u(\vec{x})$ = sign $v(\vec{x})$, i.e., the set M_0. This artifice as well as an approximate determination of the set M_0 may be carried out by means of system (12).

If the transformation A has real characteristic values, then the set M_0 of jump points is an (n - 1)-dimensional hyperplane which divides the set M into two connected sets M_+ and M_-. This hyperplane, and also the method of its construction were first discovered by A. A. Fel'dbaum (cf.(2)). We are representing it parametrically in No 5.

The second order system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + ay + u \end{aligned} \Big| \tag{14}$$

was considered by Bushaw (cf. (3)). The characteristic values of the matrix

$$\begin{Vmatrix} 0, & 1 \\ -1, & a \end{Vmatrix}$$

determine the set M and the set of jump points M_0, which in this case is a linear one lying in the phase plane x, y, as found by Bushaw.

The application of system (14) with the aid of equations (12), is an elementary problem.

In the general case, the set M_0 is a pseudomanifold and it may be approximately determined by using (12).

2. The Uniqueness Theorems. Here we prove two uniqueness theorems.

Let u_1, u_2 be two extremal controls which move a phase point along a trajectory expressed by equation (4) from a given initial position $\vec{\xi}$ towards the origin of the coordinate system, after the lapse of time t_1 and t_2, respectively. The respective trajectories (cf. formula (7)) are:

$$\vec{x}_1(t) = \vec{\varphi}_\alpha(t)\left(\xi^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau\right), \quad \vec{x}(t_1) = 0,$$

$$\vec{x}_2(t) = \vec{\varphi}_\alpha(t)\left(\xi^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau\right), \quad \vec{x}(t_2) = 0,$$

where $\vec{\varphi}_\alpha(0)\xi^\alpha = \vec{\xi}$. Then for $t_1 = t_2$, we have $u_1(t) \equiv u_2(t)$, $\vec{x}_1(t) \equiv x_2(t)$ for $0 \leqslant t \leqslant t_1 = t_2$.

Let $t_1 \leqslant t_2$ and

$$\vec{x}_1(t_1) = \vec{x}_2(t_2) = \vec{\varphi}_\alpha(t_1)\left(\xi^\alpha + \int_0^{t_1} \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau\right) = \vec{\varphi}_\alpha\ (t_2)\left(\xi^\alpha + \int_0^{t_2} \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau\right) = 0.$$

The vectors $\vec{\varphi}_\alpha(t)$, $\alpha = 1, \ldots, n$ are linearly independent for any t, and we have n equations:

$$\xi^\alpha + \int_0^{t_1} \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau = \xi^\alpha + \int_0^{t_2} \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau = 0, \quad \alpha = 1, \ldots, n,$$

whence

$$\int_0^{t_1} \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau = \int_0^{t_2} \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau, \quad \alpha = 1, \ldots, n.$$

The control $u_2(t)$ is an extremal control, hence the second equation from (12) has a solution $\vec{\psi}(t) = c_\alpha \vec{\psi}^\alpha(t)$ and $u_2(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b} = \operatorname{sign} c_\alpha \vec{\psi}^\alpha(t) \cdot \vec{b}$. Contract the second system of equations with c_α to obtain:

$$c_\alpha \int_0^{t_1} \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau = c_\alpha \int_0^{t_2} \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau = \int_0^{t_1} \vec{\psi} \cdot \vec{b}\, u_1 d\tau =$$
$$= \int_0^{t_2} \vec{\psi} \cdot \vec{b}\, u_2 d\tau = \int_0^{t_2} \vec{\psi} \cdot \vec{b} \operatorname{sign} \vec{\psi} \cdot \vec{b} = \int_0^{t_2} |\vec{\psi} \cdot \vec{b}|\, d\tau.$$

from which follows the equality

$$\int_0^{t_1} \vec{\psi} \cdot \vec{b}\, u_1 d\tau = \int_0^{t_2} |\vec{\psi} \cdot \vec{b}|\, d\tau$$

which agrees with the inequality $t_1 \leqslant t_2$ only when the conditions $t_1 = t_2$ and $u_1(t) \equiv u_2(t) \equiv \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}$ are satisfied, which therefore proves our theorem.

Therefore, we may, relying on para 1, no 4, distinguish extremals which lead to the origin of the coordinate system along optimal trajectories.

Immediately follows the statement of the Second Uniqueness Theorem.

Assume u_1, u_2 to be two optimal controls which move the phase

point along optimal trajectories as defined by equation (4) from a given initial position ξ_0 onto position $\vec{\xi}_1$ after an optimal lapse of time T. The respective optimal trajectories are denoted as below

$$\vec{x}_1(t) = \vec{\varphi}_\alpha(t)\left(\xi_0^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau\right), \quad \vec{x}_2(t) = \vec{\varphi}_\alpha(t)\left(\xi_0^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau\right),$$

where $\vec{\varphi}_\alpha(0)\xi_0^\alpha = \vec{\xi}_0$. Then for $u_1(t) \equiv u_2(t)$, $\vec{x}_1(t)$ $\vec{x}_2(t)$, $0 \leqslant t \leqslant T$.

and we have $$\vec{x}_1(T) = \vec{x}_2(T) = \vec{\varphi}_\alpha(T)\left(\xi_0^\alpha + \int_0^T \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau\right) = \vec{\varphi}_\alpha(T)\left(\xi_0^\alpha + \int_0^T \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau\right).$$

Vectors $\vec{\varphi}_\alpha(T)$, $\alpha = 1, \ldots, n$ are linearly independent, hence we have a system of n equations

$$\int_0^T \vec{\psi}^\alpha \cdot \vec{b}\, u_1 d\tau = \int_0^T \vec{\psi}^\alpha \cdot \vec{b}\, u_2 d\tau, \quad \alpha = 1, \ldots, n.$$

The end of the proof is identical with the proof of the previous theorem.

3. The Study of the Set M.

Define M(T) for all non-negative T to be the set of all phase points of the phase space X, from which one may reach the origin of the coordinate system along a trajectory defined by equation (4), by means of an admissible control, which does not necessarily have to be optimal, after a lapse of time, which is at most T. The set M is identical with the union of all M(T) for all non-negative T. We shall prove that the set M(T) is convex and closed for all positive T. (M(0) is the origin).

Assume that it is possible to arrive at the origin of the coordinate system after a lapse of time t_1 and t_2, with the motions being defined along trajectories expressed by equation (4), by means of the directions u_1 and u_2, respectively, from points ξ_1, ξ_2 respectively, which belong to the phase space X. We shall assume that $t_1 = t_2$, so in the case of an inequality $t_1 < t_2$ we shall be able to take an admissible control u_3, which is defined for all t, such that $0 \leqslant t \leqslant t_2$, by means of the equations $u_3(t) = u_1(t)$ for $0 \leqslant t \leqslant t_1$, $u_3(t) \equiv 0$ for $t_1 < t < t_2$ and such that it moves the phase point from ξ_1 onto the origin of the coordinate system after a lapse of time t_2. Therefore it is possible to arrive at the origin from any point which belongs to that interval, after that particular lapse of time $t_1 = t_2$, by means of an admissible control $\lambda u_1 + \mu u_2$, where

$$\lambda\vec{\xi}_1 + \mu\vec{\xi}_2, \quad \lambda + \mu = 1, \quad \lambda \geqslant 0, \quad \mu \geqslant 0,$$

(cf similar reasoning in para 1, no 3).

The closure of M(T) follows from the existence theorem (cf para 3, no 2). In order to prove it, take $\xi_1, \ldots, \xi_k, \ldots$ to be a sequence of points, which converge to the point ξ, which belongs to the phase space, and assume that every point $\xi_k \in M(T)$. We shall

show, that also the point $\vec{\xi} \in M(T)$.

Define u_k to be an admissible control which moves the phase point from position $\vec{\xi}_k$ onto the origin of the coordinate system, after a lapse of time $t_k \leq T$. On the basis of the existence theorem, the control u_k may be assumed to be an optimal one, such that, on replacement of an admissible control by an optimal control the time it takes to reach the origin from $\vec{\xi}_k$ may only decrease. Moreover, let us assume that $t_k \to s \leq T$. From formulas (12), $u_k(t) = \operatorname{sign} \vec{\psi}_k(t) \cdot \vec{b}$, where the function $\vec{\psi}_k(t) \cdot \vec{b}$ is not identically zero and is a solution of the nth order linear equation (8) with constant coefficients.

As previously, let us denote by $h^\alpha(t) = \vec{\psi}^\alpha(t) \cdot \vec{b}$, $\alpha = 1, \ldots, n$ the fundamental system of solutions of equation (8), and get $\vec{\psi}_k(t) \cdot \vec{b} = c_{\alpha k} h^\alpha(t)$. The coefficients $c_{\alpha k}$, $\alpha = 1, \ldots, n$, for every $k = 1, 2, \ldots$ may be assumed normalized:

$$\sum_{\alpha=1}^{n} c_{\alpha k}^2 = 1.$$

as we only care about the functions $u_k(t) = \operatorname{sign} c_{\alpha k} h^\alpha(t)$ therefore, we may assert that for $k \to \infty$ the coefficients $c_{\alpha k} \to c_\alpha$, $\alpha = 1, \ldots, n$, for $k \to \infty$ we have:

$$\vec{\psi}_k(t) \cdot \vec{b} = c_{\alpha k} h^\alpha(t) \to c_\alpha h^\alpha(t) = c_\alpha \vec{\psi}^\alpha(t) \cdot \vec{b} = \vec{\psi}(t) \cdot \vec{b}, \quad \sum_{\alpha=1}^{n} c_\alpha^2 = 1.$$

For the sequence of optimal controls $u_k(t) = \operatorname{sign} \vec{\psi}_k(t) \cdot \vec{b}$ we have for $k \to \infty$:

$$u_k(t) = \operatorname{sign} c_{\alpha k} h^\alpha(t) \to \operatorname{sign} c_\alpha h^\alpha(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b} = u(t),$$

where $\vec{\psi}(t) = c_\alpha \vec{\psi}^\alpha(t)$ is a solution of the second equation in system (12). The trajectory which corresponds to the control $u_k(t)$ (cf formula (7)) is:

$$\vec{x}_k(t) = \vec{\varphi}_\alpha(t) \left(\xi_k^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b} u_k \, d\tau \right),$$

where $\vec{\varphi}_\alpha(0) \xi^\alpha_k = \vec{\xi}_k$. The trajectory corresponding to the control $u(t)$ and the initial condition $\vec{x}(0) = \vec{\xi}$ is:

$$\vec{x}(t) = \vec{\varphi}_\alpha(t) \left(\xi^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b} u \, d\tau \right),$$

where $\vec{\varphi}_\alpha(0) \xi^\alpha = \vec{\xi}$. It follows that the control $u(t)$ moves the phase point from position $\vec{\xi}$ onto the origin of the coordinate system after a lapse of time $s \leq T$, such that we have:

$$\vec{x}(s) = \vec{x}(s) - \vec{x}_k(t_k) = \lim_{k \to \infty} (\vec{x}(s) - \vec{x}_k(s)) =$$

$$= \lim_{k \to \infty} \left(\vec{\varphi}_\alpha(s)(\xi^\alpha - \xi_k^\alpha) + \vec{\varphi}_\alpha(s) \int_0^s \vec{\psi}^\alpha \cdot \vec{b}(u - u_k) \, d\tau \right) = 0.$$

From the formula $u(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}$ and from the existence theorem for extremal controls, proved in no 2, it follows that the control $u(t)$ is optimal.

It is easy to prove that the set $M(T)$ contains interior points for arbitrary positive T. From it and the convexity and the closure of the set $M(T)$ for positive T, it follows that $M(T)$ is homeomorphic to the n-dimensional ball, and its boundary $S(T)$ is homeomorphic

to the (n - 1)-dimensional sphere.

We shall show that the topological spheres S(T) may be characterized by considering that a point which belongs to the phase space, belongs also to S(T), if and only if, when it is possible to move a phase point from it onto the origin of the coordinate system by means of an optimal control, after an optimal lapse of time T, along a trajectory defined by equation (4).

If one is able to reach the origin of the coordinate system from some point $\vec{\xi}$ by means of an admissible control, then $\vec{\xi}$ may be contained in such a neighborhood D, such that one is able to reach the origin of the coordinates from any point which belongs to D by means of some admissible control. The proof follows. Let us define u(t) to be a control which moves a phase point from position $\vec{\xi}$ onto the origin of the coordinate system, after a lapse of time T. This control, after a lapse of that time T, shall move the phase point from an arbitrary initial position $\vec{\xi}_1 \in D$ onto point $\vec{\xi}_2$, which lies in an arbitrary, previously defined neighborhood of the origin, provided that the neighborhood D is sufficiently small. But recall that the origin of the coordinate system is the interior point of the set M(T) for any positive T. Therefore, we may reach the origin from the point $\vec{\xi}_2$ by means of an admissible control, which implies, that we may reach the origin by means of an admissible control from an arbitrary point $\vec{\xi}_1 \in D$.

Consider a point $\vec{\xi} \in S(T)$. Define about it a neighborhood D which has the characteristics described above, and select in it a point sequence $\vec{\xi}_k \rightarrow \vec{\xi}$ for $k \rightarrow \infty$ and such that no $\vec{\xi}_k$ belongs to the set M(T). Define u_k to be the optimal control which moves the phase point from the position $\vec{\xi}_k$ onto the origin of the coordinate system after a lapse of time t_k; bearing in mind that $t_k > T$. Analogical reasonings to those done above, enable us to lay down a hypothesis to the effect that $u_k(t) \rightarrow u(t)$ and $t_k \rightarrow s \geqslant T$ for $k \rightarrow \infty$, and that the control u(t) is the optimal control which moves the phase point from the position $\vec{\xi}$ onto the origin of the coordinate system, after an optimal lapse of time $s \geqslant T$. As it is possible to arrive at the origin of the coordinate system by means of some admissible control, from $\vec{\xi}$ position, after a lapse of time T, it is evident that $s = T$, which proves the implication one way.

Conversely, assume that by means of an optimal control one may arrive at the origin from point $\vec{\xi}$ after an optimal lapse of time T. We shall show that $\vec{\xi} \in S(T)$. The resulting optimal trajectory is

$$\vec{x}(t) = \vec{\varphi}_\alpha(t)\left(\xi^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b}\, u\, d\tau\right).$$

The function $\vec{\psi}(t) \cdot \vec{b}$, which satisfies equation (8) is defined for all real values of the parameter t, and therefore the relay function $u(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}$ may be studied for all real values of t.

Let T_1 be an arbitrary positive number, it is evident, that one may arrive at the origin of the coordinate system from the point $\vec{x}(-T_1)$, which belongs to the phase space, after a lapse of time $T_1 + T$, by means of the optimal control

$$u(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}, \quad -T_1 \leqslant t \leqslant T,$$

where the optimal trajectory shall be

$$\vec{x}(t) = \vec{\varphi}_\alpha(t)\left(\xi^\alpha + \int_0^t \vec{\psi}^\alpha \cdot \vec{b}u d\tau\right),$$

however now, it should be studied on a much larger interval of time, namely: $-T_1 \leq t \leq T$.

Assume, that for the point ξ belonging to the interior of the set $M(T)$, for a sufficiently small $T_1 > 0$, the point

$$\vec{x}(-T_1) = \vec{\varphi}_\alpha(-T_1)\left(\xi^\alpha + \int_0^{-T_1} \vec{\psi}^\alpha \cdot \vec{b}u d\tau\right) \in M(T).$$

Because we may reach the origin of the coordinate system from $\vec{x}(-T_1)$, after a lapse of time T, and on the other hand by the above-mentioned optimal time lapse, for the motion of the point from $\vec{x}(-T_1)$ onto the origin of the coordinates, being equal to $T_1 + T > T$, which contradicts the definition of optimal time.

The obtained characteristic of the set $S(T)$ enables us to arrive at the following important observation: for $T_1 < T_2$, the sphere $S(T_1)$ is properly contained in the sphere $S(T_2)$.

The set M is obtained by taking the union of all sets $M(T)$ for all non-negative T. It follows then, that M is a convex subspace, as is every set $M(T)$ and if $\xi \in M(T)$ then the point ξ is the interior point of $M(T_2)$, where $T_2 > T_1$.

If the characteristic values of the transformation A are stable, i.e., have negative real parts, then M is identical with the phase space X. This fact is the consequence of the property that, when $T \to \infty$, the distance between the origin of the coordinate system and the set $S(T)$ also $\to \infty$. We shall prove the last theorem.

If the system (12) is solved for the initial condition $\vec{x}(0) = 0$, then in the totality of initial value $\vec{\psi}(0) = c_\alpha \vec{\psi}^\alpha(0)$, one is able to find an arbitrary covariant vector which is not equal to zero, such that in this case:

$$\vec{\psi}(0) \cdot (A\vec{x}(0) + \vec{b}u(0)) = \vec{\psi}(0) \cdot \vec{b}u(0) = \vec{\psi}(0) \cdot \vec{b} \operatorname{sign} \vec{\psi}(0) \cdot \vec{b} \geqslant 0.$$

As was stated in para 1, no 4, the vector $\vec{\psi}(0)$ may be regarded as normalized, in other words, the following equation is satisfied

$$\sum_{\alpha=1}^{n} c_\alpha^2 = \|\vec{\psi}(0)\| = 1.$$

The corresponding extremal trajectory becomes

$$\vec{x}(t) = \vec{\varphi}_\alpha(t) \int_0^t \vec{\psi}^\alpha \cdot \vec{b} \operatorname{sign} \vec{\psi} \cdot \vec{b} d\tau. \qquad (15)$$

The point $\vec{x}(t)$ shall also be investigated for negative values of the parameter t. From the Uniqueness Theorem No 2 and the Existence Theorem from para 3, No 2, for an arbitrary non-negative T, the equation (15), when investigated for the values of T lying in the interval $-T \leq t \leq 0$, gives us an optimal trajectory as per equation (4), which leads from the point $\vec{x}(-T)$ onto the origin of the coordinate system. The corresponding optimal control is $u(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}$, $-T \leq t \leq 0$, and the optimal lapse of time of motion is T. Thus, the

entire set S(T) may be obtained, if in formula (15) t is replaced by $-T$ and $\vec{\psi}(0)$ is allowed to vary in every conceivable way. As $\vec{\psi}(0)$ is assumed to be normalized, then the vector $\vec{\psi}(0)$ should describe and $(n-1)$-dimensional sphere in the space which is dual to X. (It should, however, be born in mind that the obtained map from this sphere into the topological sphere S(T) is not a homeomorphism.)

From it we obtain the following representation of the points of the set S(T):

$$\vec{x}(-T) = \vec{\varphi}_\alpha(-T)\int_0^{-T} \vec{\varphi}^\alpha \cdot \vec{b}\,\mathrm{sign}\,\vec{\varphi}\cdot\vec{b}\,d\tau = \vec{\varphi}_\alpha(-T)\int_0^{-T} \vec{\varphi}^\alpha\cdot\vec{b}\,\mathrm{sign}\,(c_\beta\vec{\varphi}^\beta\cdot\vec{b})\,d\tau,$$

where $\vec{\psi}(0) = c_\alpha\vec{\psi}^\alpha(0)$ describes an $(n-1)$-dimensional sphere.

If the transformation A possesses stable characteristic values, then the characteristic values of the transformation $-A'$ are unstable. Therefore, when $t\to-\infty$, the function $\vec{\psi}(t)\to 0$ exponentially and uniformly, relative to the initial values $\vec{\psi}(0)$, which fill the $(n-1)$-dimensional sphere; hence

$$\vec{\varphi}(-T)\cdot\vec{x}(-T) = c_\beta\vec{\varphi}^\beta(-T)\cdot\vec{\varphi}_\alpha(-T)\int_0^{-T}\vec{\varphi}^\alpha\cdot\vec{b}\,\mathrm{sign}\,\vec{\varphi}\vec{b}\,d\tau =$$

$$= \int_0^{-T} c_\alpha\vec{\varphi}^\alpha\cdot\vec{b}\,\mathrm{sign}\,(c_\alpha\vec{\varphi}^\alpha\cdot\vec{b})\,d\tau = \int_0^{-T}|\vec{\varphi}\cdot\vec{b}|\,d\tau < 0.$$

It follows that for $T\to\infty$ the expression $\vec{\psi}(-T)\cdot\vec{x}(-T)\to a$ uniformly, relative to the choice of initial conditions $\vec{\psi}(0)$, where $-\infty\leq a<0$, which means that for $T\to\infty$ the distance from an arbitrary point $\vec{x}(-T)$, which is in S(T), from the origin of the coordinate system, tends uniformly to infinity, relative to the choice of that point in the set S(T).

4. Evaluation of the function $u(\vec{x})$.

From the latter part of no 3 we have the following method of approximate evalution of the set M_o and the function $u(\vec{x})$.

We solve the system (12) for all non-positive real values of T for the initial condition $\vec{x}(0) = 0$. The initial value of $\vec{\psi}(0)$ describes the whole $(n-1)$-dimensional sphere of initial values in the dual space of X. For each choice of an initial value $\vec{\psi}(0)$, i.e., for a solution of the second equation in the system (12), we obtain a time sequence $0 > t_1 > t_2 > \ldots$, which may be infinite, and which consists of all odd multiplicity zeroes of the function $\vec{\psi}(t)\cdot\vec{b}$ for all negative real values of time, in other words, the values t_i yield all the times of jumps of the optimal control

$$u(t) = \mathrm{sign}\,\vec{\varphi}(t)\cdot\vec{b}$$

on the negative time axis. It is evident that the values $\vec{x}(t_i) \in M_o$, $i = 1, 2, \ldots$.

For the values of t, which are contained between t_i and t_{i+1}, the points $\vec{x}(t)$ belong either to the set M_+ or to the set M_-, depending on the sign of the function $u(\vec{x})$. If one performs this computation for initial values of $\vec{\psi}(0)$, which are sufficiently densely distributed on the sphere of initial values, then we may obtain an arbitrarily precise

information about the function $u(\vec{x})$ and the set M_0.

It is evident that the presented method of studying the function $u(\vec{x})$ and the set M_0, rests upon the use of a distribution of the real zeroes of solutions of the nth order linear equation (8) with constant coefficients.

$$\vec{\psi}(t)\cdot\vec{b}=c_\alpha\vec{\psi}^\alpha(t)\cdot\vec{b}=c_\alpha h^\alpha(t), \qquad \sum_{\alpha=1}^{n} c_\alpha^2=1.$$

If one wishes to investigate the general case of arbitrary complex characteristic vlues of the transformation A in the n-dimensional space, one is able to prove, that the set M_0 is an $(n-1)$-dimensional pseudomanifold, and that M_+ and M_- are connected sets. The fact, that in this general case, one fails to see clearly structure of the sets M_+, M_-, and M_0, is due to the fact that the functional relation of the real zeroes of the functions $c_\alpha h^\alpha(t)$ and the coefficients c_α, $\alpha = 1, \ldots, n$, is not a simple one. The function $c_\alpha h^\alpha(t)$ in this case is a quasi-polynomial with complex indices.

In spite of this, an approximate computation with the aid of system (12) of the above function $u(\vec{x})$ for an arbitrary equation (4) with real coefficients presents no difficulty.

If the characteristic values of the transformation A are real and distinct, then an arbitrary solution of the equation (8) assumes the form:

$$c_\alpha h^\alpha(t)=c_\alpha e^{\lambda_\alpha t},$$

where λ_α are distinct real numbers. In this case the roots of the functions $c_\alpha h^\alpha(t)$ depend in a straightforward manner on the coefficients c_α, and one may without any difficulty obtain the results stated in No 1.

The same may be stated about the zeros of solution $c_\alpha h^\alpha(t)$ of equation (8) for $n = 2$, independently of the nature of the characteristic values of the transformation A.

5. Example. Let us consider the case of real characheristic values of the transformation A. In order to make things simple, we define these characteristic values to be simple and negative. Then the set M_* in this case coincides with the space X.

We prove that the set of switch points M_0 is a hyperplane which divides the space X into two connected spaces M_+ and M_-.

An arbitrary optimal control, in our case, has the following form:

$$u(t)=\operatorname{sign} c_\alpha e^{\lambda_\alpha t},$$

where c_α, λ_α, $\alpha = 1, \ldots, n$ are distinct real numbers. For the proof we use the property of the quasi-polynomial $c_\alpha e^{\lambda_\alpha t}$, which states that for distinct c_α, λ_α, may have at most $(n-1)$ zeros, taking into account the multiplicity of the zeros, and that one may obtain an arbitrary, a priori, distribution of these zeros, by a suitable choice of the coefficients c_α, $\alpha = 1, \ldots, n$.

Let us denote by R_+^1 the following solution of the equation (4):

$$\vec{x}(t)=\vec{\varphi}_\alpha(t)\int_0^t \vec{\psi}^\alpha\cdot\vec{b}\,\operatorname{sign} e^{\lambda_1\tau}\,d\tau, \qquad -\infty<t\leqslant 0,$$

and by R^1_- the solution $\vec{x}(t) = -\vec{\varphi}_x(t) \int_0^t \vec{\varphi}^\tau \cdot \vec{b} \,\mathrm{sign}\, e^{\lambda_1 \tau} d\tau, \quad -\infty < t \leqslant 0.$

These solutions yield us optimal trajectories, which lead into the origin of the coordinate system, and the corresponding optimal controls $u_+(t) = \mathrm{sign}\, e^{\lambda_1 t} = 1, \quad u_-(t) = -\mathrm{sign}\, e^{\lambda_1 t} = -1.$

The control has no jumps along these trajectories.

The stated property about the zeros of the quasi-polynomial $c_\alpha e^{\lambda_\alpha t}$ enables us to choose the optimal control to have the form:

$$u(t) = \mathrm{sign}\,(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}),$$

such that on the interval $-\infty < t < 0$ it would have only one jump at $t = t_0$, which would be chosen beforehand, the jump being from + 1 to - 1, when going towards smaller t. Consider now a phase point which leaves the origin of the coordinate system at a time t = 0 and proceeds in the direction of negative t, along a trajectory, which corresponds to the control $u(t) = \mathrm{sign}\,(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t})$

This point shall, at first, follow R^1_+ (in the opposite direction), after the first (and only one) jump of the control u(t), which as the consequence of arbitrariness of t_0, may be made at any point along the trajectory R^1_+, the phase point then leaves the R^1_+ and never meets it again, according to the Uniqueness Theorem for the extremals (No 2).

For R^1_- we may have a symmetric construction.

As a consequence of the uniqueness of extremals, we obtain a two-dimensional surface R^2, which is the disjoint union of R^2_+ and R^2_-, by the partition by the line $R^1 = R^1_+ R^1_-$. The domain R^2_+ is filled by those parts of optimal trajectories which have their origin in the points of R^1_- and are described by a phase point, after the one and only one jump of the control $u(t) = \mathrm{sign}\,(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t})$

from + 1 to - 1 (if one proceeds in the direction of decreasing t). R^2_- is defined by the existing symmetry.

It is clear, that any point belonging to this surface, for $n \geqslant 3$ is suitable for the optimal control to have its jump at it, and so on.

In this manner we obtain a sequence of imbedded surfaces, which is totally ordered by proper inclusion:

$$R^1 \subset R^2 \subset \ldots \subset R^{n-1} \subset R^n.$$

The surface R^i has the dimensionality i and is partitioned by the surface R^{i-1} into two connected spaces R^i_- and R^i_+. The domain R^i_+ is filled by optimal trjectories which originate on R^{i-1}_-, if one proceeds in the direction of decreasing time. They correspond to the optimal controls, which are of the form:

$$u(t) = \mathrm{sign} \sum_{\alpha=1}^{i} c_\alpha e^{\lambda_\alpha t},$$

which along these trajectories assume the value + 1. The function u(t) is stabilized at this value for all non-positive t after its (i - 1)-th jump, which takes place at the time the phase point leaves the surface R_-^{i-1}.

Any point which belongs to the surfaces R_+^{n-1} or R_-^{n-1} which are subspaces of the hypersurface R^{n-1}, may be the point at which the (n - 1)-th, i.e. the last jump of the control

$$u(t) = \sum_{\alpha=1}^{n} c_\alpha e^{\lambda_\alpha t}.$$

R^n coincides with the space X, the subspaces R_+^n and R_-^n coincide with the spaces M_+ and M_- respectively.

Let the numbers $t_1, t_2, \ldots, t_{n-1}$ satisfy the condition:

$$0 \geqslant t_1 \geqslant t_2 \geqslant \ldots \geqslant t_{n-1}.$$

Then, there exists an optimal control

$$u(t_1, \ldots, t_{n-1};\ t) = \operatorname{sign} \sum_{\alpha=1}^{n} c_\alpha e^{\lambda_\alpha t},$$

such that its jumps occur for non-positive t, only at the points $t_1, \ldots, t_{n-1}$. It follows from the above, that every point which belongs to the switch hypersurface R^{n-1} may be expressed by

$$\vec{x}(t_1, \ldots, t_{n-1}) = \pm \vec{\varphi}_\alpha(t_{n-1}) \int_0^{t_{n-1}} \vec{\Phi}^\alpha \vec{b}\, u(t_1, \ldots, t_{n-1};\ \tau)\, d\tau,$$

where $0 \geqslant t_1 \geqslant t_2 \geqslant \ldots \geqslant t_{n-1}$ are suitably chosen values. Conversely, every such point $\vec{x}(t_1, \ldots, t_{n-1}) \in R^{n-1}$. From it we have the parametric representation of the hypersurface R^{n-1}:

$$\vec{x}(t_1, \ldots, t_{n-1}) = \pm \vec{\varphi}_\alpha(t_{n-1}) \int_0^{t_{n-1}} \vec{\Phi}^\alpha \cdot \vec{b}\, u(t_1, \ldots, t_{n-1};\ \tau)\, d\tau =$$
$$= \pm \vec{\varphi}_\alpha(t_{n-1}) \left(\int_0^{t_1} \vec{\Phi}^\alpha \cdot \vec{b}\, d\tau - \int_{t_1}^{t_2} \vec{\Phi}^\alpha \cdot \vec{b}\, d\tau + \ldots + (-1)^{n-2} \int_{t_{n-2}}^{t_{n-1}} \vec{\Phi}^\alpha \cdot \vec{b}\, d\tau \right), \quad (16)$$

where the parameters $t_1, \ldots, t_{n-1}$ satisfy the single condition:

$$0 \geqslant t_1 \geqslant \ldots \geqslant t_{n-1}.$$

Para 3. Several Control Parameters. Existence Theorem.

1. Generalization of the previous method, for the case of several control parameters.

The methods which have been developed in para 1, are here used without change for the case of several control parameters. Therefore, only the fundamental definitions shall be briefly stated, as well as the equations for optimal controls and their corresponding optimal trajectories.

Assume that we are given a differential vector equation with r control parameters: $u^1, \ldots, u^r$:

$$\dot{x} = A\vec{x} + \vec{b}_1 u^1 + \ldots + \vec{b}_r u^r. \qquad (17)$$

As in para 1, so here $\vec{x}$ is a vector which lies in the n-dimensional phase space X, $\vec{b}_1, \ldots, \vec{b}_r$ are the fixed vectors of this space, A is a time invariant linear transformation of X_1

The control vector function $\vec{u} = (u^1, \ldots, u^r)$ is selected from a class of piecewise continuous vector functions, such that none of their coordinates, for an arbitrary t, is numerically greater than unity, $|u^i| \leqslant 1$, $i = 1, \ldots, r$, such controls we shall call admissible.

The fundamental problem is stated as in para 1, i.e. there exist two given points $\vec{\xi}_0$, $\vec{\xi}_1$ in the phase space X, and one has to choose such an admissible vector control $\vec{u} = (u^1(t), \ldots, u^r(t))$, such that the representative point $\vec{x}(t)$ would proceed along the trajectory of equation (17), and would travel from the point $\vec{\xi}_0$ to the point $\vec{\xi}_1$ after a minimum lapse of time.

These controls, as was the case in para 1, for the optimal controls and optimal trajectories, here we give also for the non-trivial case.

The equation (17) is called non-degenerate if every i-th system $i = 1, \ldots, r$, contains n linearly independent vectors:

$$\vec{b}_i, \; A\vec{b}_i, \ldots, \; A^{n-1}\vec{b}_i, \qquad i = 1, \ldots, r.$$

Conversely, if the equation (17) is non-degenerate, then each i-th system (for i as above) contains n linearly independent functions, (cf para 1):

$$h_i^1(t) = \vec{\psi}^1(t) \cdot \vec{b}_i, \ldots, \; h_i^n(t) = \vec{\psi}^n(t) \cdot \vec{b}_i, \qquad i = 1, \ldots, r. \qquad (18)$$

The optimal controls and their corresponding optimal trajectories of the non-degenerate equation (17), which originate at the given point $\vec{\xi}_0$ which belongs to the phase space X, are defined by means of the following proposition:

All optimal controls $(u^1(t), \ldots, u^r(t))$ and their corresponding trajectories $\vec{x}(t)$, which originate at point $\vec{\xi}_0$ at time $t = 0$, by the equation (17), belong to the class of solutions of the following system of equations for the controls and their corresponding trajectories:

$$\left.\begin{array}{c} \dot{\vec{x}} = A\vec{x} + \vec{b}_1 u^1 + \ldots + \vec{b}_r u^r, \qquad \vec{x}(0) = \vec{\xi}_0 \\ \dot{\vec{\psi}} = -A'\vec{\psi}, \qquad u^i = \operatorname{sign} \vec{\psi} \cdot \vec{b}_i, \\ \vec{\psi}(0) \cdot (A\vec{x}(0) + \vec{b}_1 u^1(0) + \ldots + \vec{b}_r u^r(0)) \geqslant 0. \end{array}\right\} \qquad (19)$$

As the function (18) are linearly independent and $|\vec{\psi}(0)| \neq 0$, so the equations

$$u_i(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}_i = \operatorname{sign} c_\alpha \vec{\psi}^\alpha(t) \cdot \vec{b}_i = \operatorname{sign} c_\alpha h_i^\alpha(t), \qquad i = 1, \ldots, r,$$

uniquely define the control $(u^1(t), \ldots, u^r(t))$.

The two Uniqueness Theorems which had been proved in para 2, No 2, and which play an important role in synthesizing of optimal systems, quite naturally apply also in the case of equation (17).

2. <u>The Existence Theorem</u>. $\vec{\xi}_0$, $\vec{\xi}_1$ are two arbitrary points

in the phase space X, connected by a trajectory of equation (17), by means of some admissible control $(u^1(t), \ldots, u^r(t))$. Then there exists an optimal control, which describes the equation of motion of a phase point from the position ξ_0 onto the position ξ_1 along an optimal trajectory.

Proof*. To begin, let us solve for the class of admissible controls $(u^1(t), \ldots, u^r(t))$ in equation (17). Specificially, a control $(u^1(t), \ldots, u^r(t))$ shall be defined to be admissible, if each function $u^i(t)$, $i = 1, \ldots, r$, is measurable and everywhere, but at a finite number of points exceeds unity, i.e. $|u^i| \leqslant 1$. The ensuing solution $\vec{x}(t)$ of equation (17) is an absolutely continuous vector function, and the equation (17) is almost everywhere satisfied.

The necessary conditions (19) remain valid, for such an extension of the class of admissible controls, if one assumes that the equations (19) are satisfied almost everywhere.

In other words if $(u^1(t), \ldots, u^r(t))$ is an optimal control which belongs to the class of measurable controls, which satisfy almost everywhere the inequalities $|u^i| \leqslant 1$, $i = 1, \ldots, r$, and which describe the motion of the phase point $\vec{x}(t)$ of equation (17) from position ξ_0 onto ξ_1, along an optimal trajectory, then there exists such a solution $\vec{\psi}(t) \neq 0$ of the equation $\dot{\vec{\psi}} = -A' \vec{\psi}$, which almost everywhere

$$u^i(t) = \operatorname{sign} \vec{\psi}(t) \cdot \vec{b}_i, \quad i = 1, \ldots, r. \tag{20}$$

This may be proven by an obvious adaptation of the proof from para 1.

Let there be given $(u_k^1(t), \ldots, u_k^r(t))$, $k = 1, 2, \ldots$, a sequence of admissible (measurable controls, which describe the equation of motion of the phase point $\vec{x}(t)$ from ξ_0 onto ξ_1, along the trajectory of equation (17) and which minimize the lapse of time of travel between those points. Such a sequence shall be called the minimizing sequence.

The time of travel, which corresponds to the control $(u_k^1(t), \ldots, u_k^r(t))$, we shall denote by t_k, the lower bound of the travel time from ξ_0 to ξ_1, shall be denoted by T. As the sequence $(u_k^1(t), \ldots, u_k^r(t))$ becomes the minizing sequence, so $\lim\limits_{k\to\infty} t_k = T$.

The corresponding trajectories of equation (17), (cf. formula (7)), have the following form:

$$\vec{x}_k(t) = \vec{\varphi}_\alpha(t)\left(\xi_0^\alpha + \int_0^t \vec{\psi}^\alpha \cdot (\vec{b}_1 u_k^1 + \ldots + \vec{b}_r u_k^r)\, d\tau\right). \tag{21}$$

As $t_k \to T$ when $k \to \infty$ and $t_k \geqslant T$, the point

$$\vec{x}_k(T) = \vec{\varphi}_\alpha(T)\left(\xi_0^\alpha + \int_0^T \vec{\psi}^\alpha \cdot (\vec{b}_1 u_k^1 + \ldots + \vec{b}_r u_k^r)\, d\tau\right)$$

is defined for an arbitrary k and $\vec{x}_k(T) \to \xi_1$ for $k \to \infty$, because

* The proof given here, is considerably simpler than the proof offered previously. It was A. F. Filippov who pointed out to me the possibility of such a simplification

$(t_k) = \vec{\xi}_1$ for an arbitrary k.

We shall now, that there exists an admissible control $(u^1(t), \ldots, u^r(t))$ (in the class of measurable controls), which describes the equation of motion of the phase point from $\vec{\xi}_0$ to $\vec{\xi}_1$, during time T. This proves the existence theorem, as in agreement with the equations (20), almost everywhere

$$u^i(t) = \operatorname{sign} \vec{\Psi}(t) \cdot \vec{b}_i, \quad i = 1, \ldots, r, \quad 0 \leqslant t \leqslant T,$$

and therefore the relay control

$$(\operatorname{sign} \vec{\Psi} \cdot \vec{b}_1, \ldots, \operatorname{sign} \vec{\Psi} \cdot \vec{b}_r), \; 0 \leqslant t \leqslant T,$$

moves the phase point along the trajectory of equation (17) from the position $\vec{\xi}_0$ to $\vec{\xi}_1$ after a lapse of time T.

Let us consider functions $u_k^i(t)$, $0 \leq t \leq T$, $i = 1, \ldots, r$, $k = 1, \ldots,$ to be the elements of the Hilbert space L^2, of all square integrable functions, defined on the interval $0 \leq t \leq T$. Each function $u_k^i(t)$ is contained in a sphere of radius $\sqrt{T}$, which lies in this space, as

$$\int_0^T u_k^{i2}(t)\, dt \leqslant \int_0^T dt = T.$$

As a sphere in a Hilbert space is weakly compact, there exists such a measurable vector function $u(t) = (u^1(t), \ldots, u^r(t))$, where $u^i(t)$, $i = 1, \ldots, r$, belongs to a sphere of radius $\sqrt{T}$, such that we have a subsequence of a sequence of controls $(u_k^1(t), \ldots, u_k^r(t))$ which converges weakly to that vector function. Let this subsequence be

$$(v_k^1(t), \ldots, v_k^r(t)), \quad k = 1, 2, \ldots.$$

From the definition of weak convergence we have:

$$\int_0^T \vec{\Psi}^\alpha(t) \cdot (\vec{b}_1 v_k^1(t) + \ldots + \vec{b}_r v_k^r(t))\, dt \rightarrow$$

$$\rightarrow \int_0^T \vec{\Psi}^\alpha(t) \cdot (\vec{b}_1 u^1(t) + \ldots + \vec{b}_r u^r(t))\, dt, \quad \alpha = 1, \ldots, n,$$

for $k \rightarrow \infty$

Moreover, for $k \rightarrow \infty$

$$\vec{\Psi}_\alpha(T)\,(\xi_0^\alpha + \int_0^T \vec{\Psi}^\alpha \cdot (\vec{b}_1 v_k^1 + \ldots + \vec{b}_r v_k^r)\, dt \rightarrow \vec{\xi}_1$$

and therefore

$$\vec{\Psi}_\alpha(T)\,(\xi_0^\alpha + \int_0^T \vec{\Psi}^\alpha \cdot (\vec{b}_1 u^1 + \ldots + \vec{b}_r u^r)\, dt = \vec{\xi}_1.$$

It follows here that the measurable control $(u^1(t), \ldots, u^r(t))$ describes the equation of motion of the phase point from the position $\vec{\xi}_0$ to the position $\vec{\xi}_1$ in time T (cf formula (21)).

In order to complete the proof we have yet to show, that for $0 \leq t \leq T$, we have $|u^i| \leq 1$, $i = 1, \ldots, r$. Let us assume that this assertion is false. Let $u^1(t) > 1$ on the set $\mathfrak{S}$, whose measure is positive. Then on $\mathfrak{S}$ the following inequality is satisfied:

$$u^1(t) - u_k^1(t) \geqslant u^1(t) - 1 > 0.$$

Let f(t) be the characteristic function of the set $\mathscr{E}$, where f(t) = 1 for $t \in \mathscr{E}$, and f(t) = 0 for $t \notin \mathscr{E}$. It follows then, that:

$$\lim_{k\to\infty} \int_0^T f(t)\,(u^i(t) - v_k^i(t))\,dt \geqslant \int_0^T f(t)\,(u^i(t) - 1)\,dt =$$
$$= \int_{\mathscr{E}} (u^i(t) - 1)\,dt > 0,$$

which contradicts the assertion of weak convergence of the sequences $v_k^i(t)$ and $u_k^i(t)$.

Mathematical Institute imeni
V. A. Steklov of the USSR
Academy of Science

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